AIAA JOURNAL
Vol. 39, No. 3, March 2001

Eigensensitivity Analysis of a Defective Matrix

Zhen-yu Zhang* and Hui-sheng Zhang†
Fudan University, 200433 Shanghai, People's Republic of China

The formulas for calculating the first- to third-order perturbation coefficients of the eigenvalues and the first-order perturbation coefficients of eigenvectors of a defective matrix are derived by use of the direct perturbation method for the case where the eigenvalue problem for the first-order perturbation coefficients of the eigenvalues of the defective matrix has repeated eigenvalues. Associated with νth-order Jordan blocks of the matrix, the perturbation equations are derived by expanding the perturbed eigenvalues and eigenvectors in power series of $\eta = \varepsilon^{1/\nu}$, where ε is the small parameter of the problem, substituting the series into the perturbed eigenvalue problem and comparing the powers of η. The kth-order perturbation equations for $k = 1, \ldots, \nu - 1$ can be solved formally. The solutions contain some undetermined quantities. By use of the solvability conditions of the kth-order equations for $k = \nu, \nu + 1, \ldots$, the undetermined quantities in the solutions can be calculated in turns. When we run all orders of the Jordan blocks, the problem is solved. The numerical example shows the validity of the method.

Nomenclature

A = concerned $n \times n$ matrix
B = $n \times n$ perturbation matrix
$d_1, \ldots, d_r$ = different orders of Jordan blocks of A associated with eigenvalue λ_0 (It is assumed that $d_1 < \cdots < d_r$ when $r > 1$.)
W = matrix whose columns are all of the eigenvectors of the perturbed matrix $A + \epsilon B$ associated with d_tth-order blocks of A corresponding to λ_0
$W^{(k)}$ = kth-order perturbation coefficient matrix of W ($k \geq 1$)
ϵ = small positive perturbation parameter
Λ = diagonal matrix whose diagonal elements are all of the eigenvalues of $A + \epsilon B$ associated with d_tth-order blocks of A corresponding to λ_0
$\Lambda^{(k)}$ = kth-order perturbation coefficient matrix of Λ ($k \geq 1$)
λ_0 = concerned eigenvalue of A

Superscript

H = complex conjugate of a matrix

Introduction

THERE are many important applications of eigensensitivity analysis to dynamical analysis, identification and modification of engineering structures, vibration control and optimization. The sensitivity analysis of simple eigenvalues is easy. The sensitivity analysis of repeated eigenvalues is much more difficult, and most of the work is concentrated on the nondefective system.[1–3] There are, however, many defective systems encountered in practice. For example, when the geometric multiplicity of a nonclassical eigenvalue is less than its algebraic multiplicity the eigenvalue must be defective, and among the nonclassical modes corresponding to the eigenvalue there must be some asynchronous modes.[4] Luongo even constructed a family of defective two-degree-of-freedom systems.[5] There are only a few works on eigensensitivity analysis of defective systems because of its difficulties. Luongo solved the problem of eigensensitivity analysis of a defective matrix when the eigenvalue problems for the first-order perturbation coefficients of the eigenvalues are all simple.[6] The purpose of this paper is to solve the problem when the corresponding eigenvalue problem has repeated eigenvalues.

Formulation of the Problem

Suppose that in the Jordan canonical form of A there are s_k blocks with order d_k ($k = 1, \ldots, r$). Let $v^{(k,j)}$ and $u^{(k,j,l)}$ be respectively the left eigenvector and the lth-order principal vector (first-order principal vector is eigenvector) associated with the jth block of order d_k ($l = 1, \ldots, d_k$; $j = 1, \ldots, s_k$; $k = 1, \ldots, r$). Define

$$\tilde{A} = A - \lambda_0 I, \qquad V^{(k)} = \left[v^{(k,1)}, \ldots, v^{(k,s_k)}\right], \qquad U^{(k,0)} = 0$$

$$U^{(k,l)} = \left[u^{(k,l,1)}, \ldots, u^{(k,l,s_k)}\right], \qquad l = 1, \ldots, d_k, \qquad k = 1, \ldots, r$$

Then $V^{(k)}$ and $U^{(k,l)}$ satisfy

$$\tilde{A} U^{(k,l)} = U^{(k,l-1)}$$

$$V^{(j)^H} U^{(k,l)} = \begin{cases} I_{s_k}, & \text{when} \quad j = k \quad \text{and} \quad l = d_k \\ 0, & \text{otherwise} \end{cases}$$

$$l = 1, \ldots, d_k, \qquad j, k = 1, \ldots, r$$

Now we investigate the variations of eigenvalues and eigenvectors associated with the s_t blocks of order d_t (according to the increasing orders $t = 1, \ldots, r$) when A is perturbed by εB. In this paper it is assumed that the perturbed problem is nondefective. Thus the perturbed problem can be expressed as

$$(A + \varepsilon B) W = W \Lambda \tag{1}$$

where the columns of $W = [w^{(1)}, \ldots, w^{(s_t)}]$ are linearly independent eigenvectors of $A + \varepsilon B$ and the diagonal elements of $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{s_t})$ are the corresponding eigenvalues. Define $s_j \times s_t$ known matrix

$$Q^{(j,k,l)} = V^{(j)^H} B U^{(k,l)}, \qquad l = 1, \ldots, d_k, \qquad j, k = 1, \ldots, r$$

In this paper it is assumed that

$$\begin{vmatrix} Q^{(k,k,1)} & \cdots & Q^{(k,r,1)} \\ \cdots & \cdots & \cdots \\ Q^{(r,k,1)} & \cdots & Q^{(r,r,1)} \end{vmatrix} \neq 0, \qquad k = 1, \ldots, r \tag{2}$$

Expand W and Λ in the Puiseux series of ε[6]:

$$W = \sum_{k=0}^{\infty} W^{(k)} \eta^k, \qquad \Lambda = \sum_{k=0}^{\infty} \Lambda^{(k)} \eta^k \tag{3}$$

Received 13 September 1999; revision received 8 September 2000; accepted for publication 12 September 2000. Copyright © 2000 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

*Ph.D. Student, Fudan Institute of Mathematics; zhangzhenyu@fudan.edu.

†Professor, Department of Mechanics and Engineering Science; hszhang@fudan.edu.cn.

where

$$\eta = \varepsilon^{1/\nu}, \qquad \nu = d_t, \qquad \Lambda^{(0)} = \lambda_0 \boldsymbol{I}_{s_t}$$

$$\boldsymbol{W}^{(k)} = \left[\boldsymbol{w}^{(k,1)}, \ldots, \boldsymbol{w}^{(k,s_t)}\right], \qquad \Lambda^{(k)} = \mathrm{diag}\left[\lambda_1^{(k)}, \ldots, \lambda_{s_t}^{(k)}\right]$$

$$k = 0, 1, \ldots$$

The columns of $\boldsymbol{W}^{(0)}$ are a set of linearly independent differentiable eigenvectors of the unperturbed problem. When $k \geq 1$, the columns of $\boldsymbol{W}^{(k)}$ and the diagonal elements of $\Lambda^{(k)}$ equal to respectively $1/k!$ times of the kth-order eigenvector derivatives and of the kth-order eigenvalue derivatives with respect to parameter η. The main purpose of this paper is to calculate $\boldsymbol{W}^{(0)}$, $\boldsymbol{W}^{(k)}$ and $\Lambda^{(k)}$ ($k = 1, 2, \ldots$). In this paper it is assumed that each column of $\boldsymbol{W}^{(0)}$ is normalized so that $w_{n_j}^{(0,j)}$, the first among the components with largest absolute value in column $\boldsymbol{w}^{(0,j)}$, is 1: $w_{n_j}^{(0,j)} = 1$ ($j = 1, \ldots, s_t$). Therefore if it is stipulated that each column $\boldsymbol{w}^{(j)}$ of $\boldsymbol{W}$ is normalized so that its corresponding component is 1: $w_{n_j}^{(j)} = 1$, then when $k \geq 1$ the corresponding column $\boldsymbol{w}^{(k,j)}$ of $\boldsymbol{W}^{(k)}$ must satisfy the condition $w_{n_j}^{(k,j)} = 0$ ($j = 1, \ldots, s_t$).

Substituting Eqs. (3) into Eq. (1) and then comparing the coefficients of the powers of η, we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{W}^{(j)} = \sum_{k=1}^{j} \boldsymbol{W}^{(j-k)}\Lambda^{(k)}, \qquad j = 0, \ldots, \nu - 1 \tag{4a}$$

$$\tilde{\boldsymbol{A}}\boldsymbol{W}^{(j+\nu)} = \sum_{k=1}^{j+\nu} \boldsymbol{W}^{(j+\nu-k)}\Lambda^{(k)} - \boldsymbol{B}\boldsymbol{W}^{(j)}, \qquad j = 0, 1, \ldots \tag{4b}$$

Define

$$\boldsymbol{C}^{(1,j,k)} = \alpha^{(j,\nu+1-d_k)}\Lambda^{(1)^{d_k}} + d_k\alpha^{(j,\nu-d_k)}\Lambda^{(1)^{d_k-1}}\Lambda^{(2)}$$

$$\boldsymbol{C}^{(2,j,k)} = \alpha^{(j,\nu+2-d_k)}\Lambda^{(1)^{d_k}} + d_k\alpha^{(j,\nu+1-d_k)}\Lambda^{(1)^{d_k-1}}\Lambda^{(2)} + \alpha^{(j,\nu-d_k)}\left[d_k\Lambda^{(1)^{d_k-1}}\Lambda^{(3)} + \frac{d_k(d_k-1)}{2}\Lambda^{(1)^{d_k-2}}\Lambda^{(2)^2}\right]$$

$$\boldsymbol{D}^{(j,p)} = \sum_{\substack{h_1,\ldots,h_j \geq 1 \\ h_1+\cdots+h_j = p}} \prod_{q=1}^{j} \Lambda^{(h_q)}, \qquad \boldsymbol{E}^{(m,l,k,j)} = \sum_{p=j}^{m} \alpha^{(k,l-p)}\boldsymbol{D}^{(j,p)}$$

$$\boldsymbol{G}^{(l)} = \sum_{k=1}^{t-1}\sum_{j=1}^{d_k-1} \boldsymbol{U}^{(k,j)}\boldsymbol{E}^{(d_k+l,\nu+l,k,j)} + \sum_{k=t}^{r}\sum_{j=1}^{\nu-1} \boldsymbol{U}^{(k,j)}\boldsymbol{E}^{(\nu+l,\nu+l,k,j)}$$

$$\tilde{\boldsymbol{G}}^{(l)} = \sum_{k=1}^{t-1}\sum_{j=1}^{d_k-1} \boldsymbol{U}^{(k,j+1)}\boldsymbol{E}^{(d_k+l,\nu+l,k,j)} + \sum_{k=t}^{r}\sum_{j=1}^{\nu-1} \boldsymbol{U}^{(k,j+1)}\boldsymbol{E}^{(\nu+l,\nu+l,k,j)}$$

In the preceding equations $\alpha^{(k,j)} = [\alpha_{ml}^{(k,j)}]$ is a $s_k \times s_t$ to-be-determined coefficients' matrix ($k = 1, \ldots, r$; $j = 0, 1, \ldots$). It follows from Eqs. (4a) that

$$\boldsymbol{W}^{(0)} = \sum_{k=t}^{r} \boldsymbol{U}^{(k,1)}\alpha^{(k,0)} \tag{4c}$$

$$\boldsymbol{W}^{(l)} = \sum_{k=1}^{t-1}\sum_{j=1}^{l+d_k-\nu} \boldsymbol{U}^{(k,j+1)}\boldsymbol{E}^{(l,l,k,j)} + \sum_{k=t}^{r}\sum_{j=1}^{l} \boldsymbol{U}^{(k,j+1)}\boldsymbol{E}^{(l,l,k,j)} + \sum_{\substack{k=\min(m) \\ l+d_m-\mu \geq 0}}^{r} \boldsymbol{U}^{(k,1)}\alpha^{(k,l)}, \qquad l = 1, \ldots, \nu - 1 \tag{4d}$$

Substituting Eqs. (4c) and (4d) into the equation for $\boldsymbol{W}^{(\nu)}$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{W}^{(\nu)} = \sum_{k=1}^{t-1} \boldsymbol{U}^{(k,d_k)}\alpha^{(k,\nu-d_k)}\Lambda^{(1)^{d_k}} + \sum_{k=t}^{r} \boldsymbol{U}^{(k,\nu)}\alpha^{(k,0)}\Lambda^{(1)^{\nu}} - \boldsymbol{B}\boldsymbol{W}^{(0)} + \boldsymbol{G}^{(0)} \stackrel{\text{def}}{=} \boldsymbol{R}^{(0)} + \boldsymbol{G}^{(0)} \tag{5}$$

On the right-hand side of Eq. (5), all of the terms except $\boldsymbol{R}^{(0)}$ are in the range of $\tilde{\boldsymbol{A}}[\Re(\tilde{\boldsymbol{A}})]$. From the solvability condition of Eq. (5), $\boldsymbol{V}^{(j)^H}\boldsymbol{R}^{(0)} = 0$ ($j = 1, \ldots, r$), it can be obtained that

$$\sum_{k=t}^{r} \boldsymbol{Q}^{(j,k,1)}\alpha^{(k,0)} = \alpha^{(j,\nu-d_j)}\Lambda^{(1)^{d_j}}, \qquad j = 1, \ldots, t \tag{6a}$$

$$\sum_{k=t}^{r} \boldsymbol{Q}^{(j,k,1)}\alpha^{(k,0)} = 0, \qquad j = t+1, \ldots, r \tag{6b}$$

It follows from Eqs. (6b) that

$$\alpha^{(k,0)} = \boldsymbol{Q}^{(k)}\alpha^{(t,0)}, \qquad k = t+1, \ldots, r \tag{7}$$

where $\boldsymbol{Q}^{(t+1)}, \ldots, \boldsymbol{Q}^{(r)}$ can be determined by

$$\begin{bmatrix} \boldsymbol{Q}^{(t+1)} \\ \vdots \\ \boldsymbol{Q}^{(r)} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{Q}^{(t+1,t+1,1)} & \cdots & \boldsymbol{Q}^{(t+1,r,1)} \\ \cdots & \cdots & \cdots \\ \boldsymbol{Q}^{(r,t+1,1)} & \cdots & \boldsymbol{Q}^{(r,r,1)} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{Q}^{(t+1,t,1)} \\ \vdots \\ \boldsymbol{Q}^{(r,t,1)} \end{bmatrix} \stackrel{\text{def}}{=} -\boldsymbol{Q}\begin{bmatrix} \boldsymbol{Q}^{(t+1,t,1)} \\ \vdots \\ \boldsymbol{Q}^{(r,t,1)} \end{bmatrix}$$

Substituting Eqs. (7) into the tth equation in Eqs. (6a), we obtain the following eigenvalue problem with the columns of $\alpha^{(t,0)}$ as its eigenvectors:

$$\left[\boldsymbol{Q}^{(t,t,1)} + \sum_{k=t+1}^{r} \boldsymbol{Q}^{(t,k,1)}\boldsymbol{Q}^{(k)}\right]\alpha^{(t,0)} = \alpha^{(t,0)}\Lambda^{(1)^{\nu}} \tag{8}$$

It can be seen from Eqs. (2) and (6) that for any t, $\Lambda^{(1)}$ is nonsingular (cf. Appendix). The problem was solved by Luongo[6] when all of the eigenvalues of problem (8) are simple. The task of this work is to solve the problem when Eq. (8) has repeated eigenvalues.

First- and Second-Order Eigenvalue Perturbation Coefficients and the Differentiable Eigenvectors

Let $\mu_1^{\nu}, \ldots, \mu_e^{\nu}$ be the different eigenvalues of problem (8) with multiplicities $m_1, \ldots, m_e$, respectively. Then $\Lambda^{(1)} = \mathrm{diag}(\mu_1\boldsymbol{I}_{m_1}, \ldots, \mu_e\boldsymbol{I}_{m_e})$. Take arbitraryly m_s of linearly independent eigenvectors of problem (8) corresponding to μ_s and construct a matrix $\beta^{(s)} = [\beta_{jk}^{(s)}]$ with these eigenvectors as its columns ($s = 1, \ldots, e$). Let $\alpha^{(t,0)}$ and $\boldsymbol{W}^{(0)}$ be partitioned correspondingly into blocks

$$\alpha^{(t,0)} = \left[\alpha^{(t,0,1)}, \ldots, \alpha^{(t,0,e)}\right], \qquad \boldsymbol{W}^{(0)} = \left[\boldsymbol{W}^{(0,1)}, \ldots, \boldsymbol{W}^{(0,e)}\right]$$

Then

$$\alpha^{(t,0,s)} = \beta^{(s)}\gamma^{(s)}$$

$$\boldsymbol{W}^{(0,s)} = \left[\boldsymbol{U}^{(t,1)} + \sum_{k=t+1}^{r} \boldsymbol{U}^{(k,1)}\boldsymbol{Q}^{(k)}\right]\beta^{(s)}\gamma^{(s)} \stackrel{\text{def}}{=} \hat{\boldsymbol{Q}}\beta^{(s)}\gamma^{(s)}$$

where $\gamma^{(s)} = [\gamma_{jk}^{(s)}]$ is an $m_s \times m_s$ to-be-determined matrix ($s = 1, \ldots, e$). Define the following known matrices:

$$\beta = \left[\beta^{(1)}, \ldots, \beta^{(e)}\right], \qquad \beta^{-T} = \left[\delta^{(1)}, \ldots, \delta^{(e)}\right]$$

$$\boldsymbol{f}^{(0)} = \boldsymbol{U}^{(t,\nu)} + \sum_{k=t+1}^{r} \boldsymbol{U}^{(k,\nu)}\boldsymbol{Q}^{(k)}$$

$$\boldsymbol{f}^{(1)} = \sum_{l=1}^{t-1}\boldsymbol{U}^{(l,d_l)}\left[\boldsymbol{Q}^{(l,t,1)} + \sum_{k=t+1}^{r}\boldsymbol{Q}^{(l,k,1)}\boldsymbol{Q}^{(k)}\right] - \boldsymbol{B}\hat{\boldsymbol{Q}}$$

It follows from Eqs. (7) that

$$\boldsymbol{R}^{(0)} = \boldsymbol{f}^{(0)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)^{\nu}} + \boldsymbol{f}^{(1)}\alpha^{(t,0)} \tag{8a}$$

Let $\tilde{\boldsymbol{A}}^{(1)}$ be any generalized {1}-inverse of $\tilde{\boldsymbol{A}}$ and define $\tilde{\boldsymbol{W}}^{(\nu)} = \tilde{\boldsymbol{A}}^{(1)}\boldsymbol{R}^{(0)}$. Then the general solution of Eq. (5) can be expressed as

$$\boldsymbol{W}^{(\nu)} = \tilde{\boldsymbol{W}}^{(\nu)} + \tilde{\boldsymbol{G}}^{(0)} + \sum_{k=1}^{r}\boldsymbol{U}^{(k,1)}\alpha^{(k,\nu)}$$

Substituting the preceding results into the equation for $\boldsymbol{W}^{(\nu+1)}$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{W}^{(\nu+1)} = \tilde{\boldsymbol{W}}^{(\nu)}\boldsymbol{\Lambda}^{(1)} + \sum_{k=1}^{t-1}\boldsymbol{U}^{(k,d_k)}\boldsymbol{C}^{(1,k,k)} + \sum_{k=t}^{r}\boldsymbol{U}^{(k,\nu)}\boldsymbol{C}^{(1,k,t)}$$

$$-\boldsymbol{B}\boldsymbol{W}^{(1)} + \boldsymbol{G}^{(1)} \stackrel{\text{def}}{=} \boldsymbol{R}^{(1)} + \boldsymbol{G}^{(1)} \tag{9}$$

All of the terms except $\boldsymbol{R}^{(1)}$ on the right-hand side of Eq. (9) are in $\Re(\tilde{\boldsymbol{A}})$. Define

$$\boldsymbol{\Phi}^{(1,j)} = \begin{cases} \boldsymbol{V}^{(j)^H}\boldsymbol{B}\tilde{\boldsymbol{W}}^{(1)}, & \nu = 1 \\ \sum_{k=t}^{r}\boldsymbol{Q}^{(j,k,2)}\alpha^{(k,0)}\boldsymbol{\Lambda}^{(1)} + \boldsymbol{Q}^{(j,t-1,1)}\alpha^{(t-1,1)} & t > 1, \quad \nu = d_{t-1}+1 \\ \sum_{k=t}^{r}\boldsymbol{Q}^{(j,k,2)}\alpha^{(k,0)}\boldsymbol{\Lambda}^{(1)}, & \text{otherwise} \end{cases}$$

$$j = 1, \ldots, r$$

From the solvability condition of Eq. (9), it can be obtained that

$$\sum_{k=t}^{r}\boldsymbol{Q}^{(j,k,1)}\alpha^{(k,1)} + \boldsymbol{\Phi}^{(1,j)} = \boldsymbol{C}^{(1,j,j)} + \boldsymbol{V}^{(j)^H}\tilde{\boldsymbol{W}}^{(\nu)}\boldsymbol{\Lambda}^{(1)}$$

$$j = 1, \ldots, t \tag{10a}$$

$$\sum_{k=t}^{r}\boldsymbol{Q}^{(j,k,1)}\alpha^{(k,1)} + \boldsymbol{\Phi}^{(1,j)} = \boldsymbol{V}^{(j)^H}\tilde{\boldsymbol{W}}^{(\nu)}\boldsymbol{\Lambda}^{(1)}$$

$$j = t+1, \ldots, r \tag{10b}$$

Define the following known quantities:

$$\boldsymbol{f}^{(0,j)} = \boldsymbol{V}^{(j)^H}\tilde{\boldsymbol{A}}^{(1)}\boldsymbol{f}^{(0)}$$

$$\boldsymbol{f}^{(1,j)} = \begin{cases} \boldsymbol{V}^{(j)^H}\left[\tilde{\boldsymbol{A}}^{(1)}\boldsymbol{f}^{(1)} - \boldsymbol{B}\tilde{\boldsymbol{A}}^{(1)}\boldsymbol{f}^{(0)}\right], & \nu = 1 \\ \boldsymbol{V}^{(j)^H}\tilde{\boldsymbol{A}}^{(1)}\boldsymbol{f}^{(1)} - \left[(\boldsymbol{Q}^{(j,t,2)} + \sum_{k=t+1}^{r}\boldsymbol{Q}^{(j,k,2)}\boldsymbol{Q}^{(k)}\right] & \nu > 1 \end{cases}$$

$$\boldsymbol{f}^{(2,j)} = \begin{cases} -\boldsymbol{V}^{(j)^H}\boldsymbol{B}\tilde{\boldsymbol{A}}^{(1)}\boldsymbol{f}^{(1)}, & \nu = 1 \\ -\boldsymbol{Q}^{(j,t-1,1)}\left[\boldsymbol{Q}^{(t-1,t,1)} + \sum_{k=t+1}^{r}\boldsymbol{Q}^{(t-1,k,1)}\boldsymbol{Q}^{(k)}\right], & t > 1 \text{ and } \nu = d_{t-1}+1 \\ 0, & \text{otherwise} \end{cases}$$

$$j = 1, \ldots, r$$

$$\begin{bmatrix} \boldsymbol{g}^{(j,t+1)} \\ \vdots \\ \boldsymbol{g}^{(j,r)} \end{bmatrix} = \boldsymbol{Q}\begin{bmatrix} \boldsymbol{f}^{(j,t+1)} \\ \vdots \\ \boldsymbol{f}^{(j,r)} \end{bmatrix}$$

$$\boldsymbol{g}^{(j)} = \sum_{k=t+1}^{r}\boldsymbol{Q}^{(t,k,1)}\boldsymbol{g}^{(j,k)} - \boldsymbol{f}^{(j,t)}, \qquad j = 0, 1, 2$$

It follows from Eqs. (10b) that

$$\alpha^{(k,1)} = \boldsymbol{Q}^{(k)}\alpha^{(t,1)} + \boldsymbol{g}^{(0,k)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)^{\nu+1}} + \boldsymbol{g}^{(1,k)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)}$$

$$+\boldsymbol{g}^{(2,k)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)^{-\nu+1}} \stackrel{\text{def}}{=} \boldsymbol{Q}^{(k)}\alpha^{(t,1)} + \boldsymbol{\Psi}^{(k)}$$

$$k = t+1, \ldots, r \tag{11}$$

Define

$$\tilde{\alpha}^{(t,1)} = \alpha^{(t,0)^{-1}}\alpha^{(t,1)} = \left[\tilde{\alpha}^{(t,1,j,k)}\right]$$

where the $m_j \times m_k$ matrix $\tilde{\alpha}^{(t,1,j,k)} = [\tilde{\alpha}_{ml}^{(t,1,j,k)}]$ is the (j,k)th block after $\tilde{\alpha}^{(t,1)}$ is partitioned correspondingly. Substituting Eqs. (11) into the tth equation in Eqs. (10a) and using Eq. (8), we obtain

$$\boldsymbol{\Lambda}^{(1)^{\nu}}\tilde{\alpha}^{(t,1)} - \tilde{\alpha}^{(t,1)}\boldsymbol{\Lambda}^{(1)^{\nu}} + \boldsymbol{X} = \nu\boldsymbol{\Lambda}^{(1)^{\nu-1}}\boldsymbol{\Lambda}^{(2)} \tag{12}$$

where

$$\boldsymbol{X} = \alpha^{(t,0)^{-1}}\left[\boldsymbol{g}^{(0)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)^{\nu+1}} + \boldsymbol{g}^{(1)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)} + \boldsymbol{g}^{(2)}\alpha^{(t,0)}\boldsymbol{\Lambda}^{(1)^{-\nu+1}}\right] \tag{12a}$$

Comparing the diagonal blocks of Eq. (12), we obtain the following eigenvalue problem with columns of $\gamma^{(k)}$ as its eigenvectors

$$\gamma^{(k)^{-1}}\left\{(1/\nu)\delta^{(k)^T}\left[\mu_k^2\boldsymbol{g}^{(0)} + \mu_k^{2-\nu}\boldsymbol{g}^{(1)} + \mu_k^{2-2\nu}\boldsymbol{g}^{(2)}\right]\beta^{(k)}\right\}\gamma^{(k)} = \boldsymbol{\Lambda}^{(2,k)}$$

$$k = 1, \ldots, e \tag{13}$$

For simplicity it is assumed in this paper that for any k $(1 \le k \le e)$, the eigenvalues of problem (13) are all simple. Take arbitrarily m_k linearly independent eigenvectors of problem (13) and construct an $m_k \times m_k$ matrix $\hat{\gamma}^{(k)}$ with these eigenvectors as its columns $(k = 1, \ldots, e)$. Define the following known matrices:

$$\hat{\alpha}^{(t,0)} = \left[\hat{\alpha}_{ml}^{(t,0)}\right] = \left[\beta^{(1)}\hat{\gamma}^{(1)}, \ldots, \beta^{(e)}\hat{\gamma}^{(e)}\right], \qquad \hat{\boldsymbol{W}}^{(0)} = \hat{Q}\hat{\alpha}^{(t,0)}$$

In $\hat{\boldsymbol{w}}^{(0,j)}$, the column of $\hat{\boldsymbol{W}}^{(0)}$, if the first among the components with largest absolute value is $\hat{w}_{n_j}^{(0,j)}$, then the corresponding column of $\boldsymbol{W}^{(0)}$ is $\boldsymbol{w}^{(0,j)} = \hat{\boldsymbol{w}}^{(0,j)}/\hat{w}_{n_j}^{(0,j)}$, and the elements of $\alpha^{(t,0)}$ are $\alpha_{mj}^{(t,0)} = \hat{\alpha}_{mj}^{(t,0)}/\hat{w}_{n_j}^{(0,j)}$ $(m, j = 1, \ldots, s_t)$. Thus $\boldsymbol{W}^{(0)}, \alpha^{(t,0)}$ are calculated. Then $\alpha^{(k,0)}$ $(k = t+1, \ldots, r)$ can be calculated by using Eqs. (7). Then $\alpha^{(j,\nu-d_j)}$ $(j = 1, \ldots, t-1)$ can be calculated from Eqs. (6a). Thus $\boldsymbol{R}^{(0)}$ can be calculated from Eq. (8a), and then $\tilde{\boldsymbol{W}}^{(\nu)} = \tilde{\boldsymbol{A}}^{(1)}\boldsymbol{R}^{(0)}$ can be obtained.

Third-Order Eigenvalue Perturbation Coefficients and the First-Order Eigenvector Perturbation Coefficients

$\boldsymbol{X}$ can be calculated by using Eq. (12a). Let $\boldsymbol{X}$ be partitioned correspondingly, $\boldsymbol{X} = [\boldsymbol{X}^{(j,k)}]$. The nondiagonal blocks of $\tilde{\alpha}^{(t,1)}$ can be obtained by comparing the nondiagonal blocks of Eq. (12):

$$\tilde{\alpha}^{(t,1,j,k)} = \frac{\boldsymbol{X}^{(j,k)}}{\mu_k^{\nu} - \mu_j^{\nu}}, \qquad j \ne k; \qquad j, k = 1, \ldots, e$$

Define the following known quantities:

$$F^{(1,j)} = \sum_{k=t+1}^{r} Q^{(j,k,1)}\Psi^{(k)} + \Phi^{(1,j)} - d_j\alpha^{(j,\nu-d_j)}\Lambda^{(1)^{d_j-1}}\Lambda^{(2)}$$

$$- V^{(j)^H}\tilde{W}^{(\nu)}\Lambda^{(1)}, \qquad j = 1,\dots,t-1$$

$$\overset{*}{W}^{(1)} = \begin{cases} \tilde{W}^{(1)}, & \nu = 1 \\ \sum_{k=t}^{r} U^{(k,2)}\alpha^{(k,0)}\Lambda^{(1)} \\ \quad + U^{(t-1,1)}\alpha^{(t-1,1)}, & t > 1 \quad \text{and} \quad \nu = d_{t-1}+1 \\ \sum_{k=t}^{r} U^{(k,2)}\alpha^{(k,0)}\Lambda^{(1)}, & \text{otherwise} \end{cases}$$

$$F^{(1)} = \tilde{W}^{(\nu)}\Lambda^{(1)} - B\overset{*}{W}{}^{(1)} + \nu f^{(0)}\alpha^{(t,0)}\Lambda^{(1)^{\nu-1}}\Lambda^{(2)}$$

$$+ \sum_{k=1}^{t-1} U^{(k,d_k)}\left[F^{(1,k)} + d_k\alpha^{(k,\nu-d_k)}\Lambda^{(1)^{d_k-1}}\Lambda^{(2)}\right]$$

$$+ \sum_{k=t+1}^{r}\left[U^{(k,\nu)}\Psi^{(k)}\Lambda^{(1)^{(\nu)}} - BU^{(k,1)}\Psi^{(k)}\right]$$

It follows from Eqs. (10a) that

$$R^{(1)} = f^{(0)}\alpha^{(t,1)}\Lambda^{(1)^{\nu}} + f^{(1)}\alpha^{(t,1)} + F^{(1)} \tag{14}$$

Define $\tilde{W}^{(\nu+1)} = \tilde{A}^{(1)}R^{(1)}$. Then the general solution of Eq. (9) can be expressed as

$$W^{(\nu+1)} = \tilde{W}^{(\nu+1)} + \tilde{G}^{(1)} + \sum_{k=1}^{r} U^{(k,1)}\alpha^{(k,\nu+1)}$$

Substituting the preceding results into the equation for $W^{(\nu+2)}$ in Eqs. (4b), we obtain

$$\tilde{A}W^{(\nu+2)} = \sum_{k=1}^{t-1} U^{(k,d_k)}C^{(2,k,k)} + \sum_{k=t}^{r} U^{(k,\nu)}C^{(2,k,t)} + \tilde{W}^{(\nu+1)}\Lambda^{(1)}$$

$$+ \tilde{W}^{(\nu)}\Lambda^{(2)} - BW^{(2)} + G^{(2)} \overset{\text{def}}{=} R^{(2)} + G^{(2)} \tag{15}$$

All of the terms except $R^{(2)}$ on the right-hand side of Eq. (15) are in $\Re(\tilde{A})$. Define

$$T^{(0,j)} = \begin{cases} V^{(j)^H}B\tilde{W}^{(2)}, & \nu \le 2 \\ \sum_{k=t}^{r} Q^{(j,k,3)}\alpha^{(k,0)}\Lambda^{(1)^2}, & \nu > 2 \end{cases}$$

$$\tilde{T}^{(0,j)} = \begin{cases} T^{(0,j)}, & \nu = 1 \\ T^{(0,j)} + \sum_{k=t}^{r} Q^{(j,k,2)}\left[\alpha^{(k,1)}\Lambda^{(1)} + \alpha^{(k,0)}\Lambda^{(2)}\right], & \nu > 1 \end{cases}$$

$$\Phi^{(2,j)} = \begin{cases} \tilde{T}^{(0,j)} + Q^{(j,t-1,1)}\alpha^{(t-1,2)}, & \nu = 2 \quad \text{and} \quad t = 2 \\ & \text{or} \quad t > 1 \quad \text{and} \\ & \nu = d_{t-1}+2 \\ \tilde{T}^{(0,j)} + Q^{(j,t-1,2)}\alpha^{(t-1,1)}\Lambda^{(1)} & t = 2 \quad \text{and} \\ \quad + Q^{(j,t-1,1)}\alpha^{(t-1,2)}, & \nu = d_{t-1}+1 > 2 \\ & \text{or } t > 2 \quad \text{and} \\ & \nu = d_{t-1}+1 \\ & > d_{t-2}+2 \\ \tilde{T}^{(0,j)} + Q^{(j,t-1,2)}\alpha^{(t-1,1)}\Lambda^{(1)} & t > 2, \quad \text{and} \\ \quad + \sum_{k=t-2}^{t-1} Q^{(j,k,1)}\alpha^{(k,2)}, & \nu = d_{t-1}+1 \\ & = d_{t-2}+2 \\ \tilde{T}^{(0,j)}, & \text{otherwise} \end{cases}$$

$$j = 1,\dots,r$$

Define the following known quantities:

$$T^{(1,j)} = V^{(j)^H}\left[\tilde{A}^{(1)}F^{(1)}\Lambda^{(1)} + \tilde{W}^{(\nu)}\Lambda^{(2)}\right]$$

$$\hat{T}^{(1,j)} = \begin{cases} T^{(1,j)}, & \nu = 1 \\ T^{(1,j)} - \sum_{k=t+1}^{r} Q^{(j,k,2)} \\ \quad \times\left[\Psi^{(k)}\Lambda^{(1)} + \alpha^{(k,0)}\Lambda^{(2)}\right] \\ \quad - Q^{(j,t,2)}\alpha^{(t,0)}\Lambda^{(2)}, & \nu > 1 \end{cases}$$

$$\tilde{T}^{(1,j)} = \begin{cases} \hat{T}^{(1,j)} - V^{(j)^H}B\tilde{A}^{(1)}F^{(1)}, & \nu = 1 \\ \hat{T}^{(1,j)} - V^{(j)^H}B\tilde{W}^{(2)}, & \nu = 2 \\ \hat{T}^{(1,j)} - \sum_{k=t}^{r} Q^{(j,k,3)}\alpha^{(k,0)}\Lambda^{(1)^2}, & \nu > 2 \end{cases}$$

$$\tilde{F}^{(2,j)} =$$

$$\begin{cases} \tilde{T}^{(1,j)} - Q^{(j,t-1,1)}\alpha^{(t-1,2)}, & t > 1 \ \text{and} \ \nu = d_{t-1}+2 \\ \tilde{T}^{(1,j)} - Q^{(j,t-1,2)}\alpha^{(t-1,1)}\Lambda^{(1)} & t = 2 \ \text{and} \ \nu = d_{t-1}+1 > 2 \\ \quad - Q^{(j,t-1,1)}F^{(1,t-1)}\Lambda^{(1)^{1-\nu}}, & \text{or } t > 2 \ \text{and} \\ & \nu = d_{t-1}+1 > d_{t-2}+2 \\ \tilde{T}^{(1,j)} - Q^{(j,t-1,2)}\alpha^{(t-1,1)}\Lambda^{(1)} \\ \quad - Q^{(j,t-1,1)}F^{(1,t-1)}\Lambda^{(1)^{1-\nu}} & t > 2 \ \text{and} \\ \quad - Q^{(j,t-2,1)}\alpha^{(t-2,2)}, & \nu = d_{t-1}+1 = d_{t-2}+2 \\ \tilde{T}^{(1,j)} - Q^{(j,t-1,1)}F^{(1,t-1)}\Lambda^{(1)^{1-\nu}}, & \nu = 2 \ \text{and} \ t = 2 \\ \tilde{T}^{(1,j)}, & \text{otherwise} \end{cases}$$

$$j = t,\dots,r$$

$$\begin{bmatrix} F^{(2,t+1)} \\ \vdots \\ F^{(2,r)} \end{bmatrix} = Q\begin{bmatrix} \tilde{F}^{(2,t+1)} \\ \vdots \\ \tilde{F}^{(2,r)} \end{bmatrix}$$

From the solvability condition of Eq. (15), the following can be obtained:

$$\sum_{k=t}^{r} Q^{(j,k,1)}\alpha^{(k,2)} + \Phi^{(2,j)}$$

$$= C^{(2,j,j)} + V^{(j)^H}\left[\tilde{W}^{(\nu+1)}\Lambda^{(1)} + \tilde{W}^{(\nu)}\Lambda^{(2)}\right], \qquad j = 1,\dots,t \tag{16a}$$

$$\sum_{k=t}^{r} Q^{(j,k,1)}\alpha^{(k,2)} + \Phi^{(2,j)} = V^{(j)^H}\left[\tilde{W}^{(\nu+1)}\Lambda^{(1)} + \tilde{W}^{(\nu)}\Lambda^{(2)}\right]$$

$$j = t+1,\dots,r \tag{16b}$$

It follows from Eqs. (16b) that

$$\alpha^{(k,2)} = Q^{(k)}\alpha^{(t,2)} + g^{(0,k)}\alpha^{(t,1)}\Lambda^{(1)^{\nu+1}} + g^{(1,k)}\alpha^{(t,1)}\Lambda^{(1)}$$

$$+ g^{(2,k)}\alpha^{(t,1)}\Lambda^{(1)^{-\nu+1}} + F^{(2,k)}, \qquad k = t+1,\dots,r \tag{17}$$

Define the known matrix

$$\tilde{F}^{(3)} = \sum_{k=t+1}^{r} Q^{(t,k,1)}F^{(2,k)} - \frac{\nu(\nu-1)}{2}\alpha^{(t,0)}\Lambda^{(1)^{\nu-2}}\Lambda^{(2)^2} - \tilde{F}^{(2,t)}$$

Substituting Eqs. (17) into the tth equation in Eqs. (16a) and using Eq. (8), we obtain

$$\Lambda^{(1)^{\nu}}\tilde{\alpha}^{(t,2)} - \tilde{\alpha}^{(t,2)}\Lambda^{(1)^{\nu}} + \alpha^{(t,0)^{-1}}\left[g^{(0)}\alpha^{(t,0)}\tilde{\alpha}^{(t,1)}\Lambda^{(1)^{\nu+1}}\right.$$

$$\left. + g^{(1)}\alpha^{(t,0)}\tilde{\alpha}^{(t,1)}\Lambda^{(1)} + g^{(2)}\alpha^{(t,0)}\tilde{\alpha}^{(t,1)}\Lambda^{(1)^{-\nu+1}}\right]$$

$$- \nu\tilde{\alpha}^{(t,1)}\Lambda^{(1)^{\nu-1}}\Lambda^{(2)} + \alpha^{(t,0)^{-1}}\tilde{F}^{(3)} = \nu\Lambda^{(1)^{\nu-1}}\Lambda^{(3)} \tag{18}$$

where $\tilde{\alpha}^{(t,2)} = \alpha^{(t,0)^{-1}}\alpha^{(t,2)}$. Define the known matrix

$$\boldsymbol{F}^{(3)} = (1/\nu)\alpha^{(t,0)^{-1}}\tilde{\boldsymbol{F}}^{(3)}\boldsymbol{\Lambda}^{(1)^{1-\nu}}$$

Let $\boldsymbol{F}^{(3)}$ be partitioned correspondingly, $\boldsymbol{F}^{(3)} = [\boldsymbol{F}^{(3,j,k)}]$. Define the known matrices

$$\boldsymbol{Y}^{(k)} = \left[y_{ml}^{(k)}\right] = \boldsymbol{F}^{(3,k,k)} + \frac{1}{\nu}\gamma^{(k)^{-1}}\delta^{(k)^T} \times \left[\mu_k^2\boldsymbol{g}^{(0)} + \mu_k^{2-\nu}\boldsymbol{g}^{(1)} + \mu_k^{2-2\nu}\boldsymbol{g}^{(2)}\right]\sum_{\substack{1\le l\le e\\ l\ne k}}\beta^{(l)}\gamma^{(l)}\tilde{\alpha}^{(t,1,l,k)}$$

$$k = 1,\ldots,e$$

Comparing the diagonal blocks in Eq. (18) and using Eqs. (13), we obtain

$$\boldsymbol{\Lambda}^{(2,k)}\tilde{\alpha}^{(t,1,k,k)} - \tilde{\alpha}^{(t,1,k,k)}\boldsymbol{\Lambda}^{(2,k)} + \boldsymbol{Y}^{(k)} = \boldsymbol{\Lambda}^{(3,k)}, \qquad k = 1,\ldots,e \tag{19}$$

$\boldsymbol{\Lambda}^{(3)}$ can be obtained by comparing the diagonal elements in Eqs. (19):

$$\lambda_j^{(3,k)} = y_{jj}^{(k)}, \qquad j = 1,\ldots,m_k, \qquad k = 1,\ldots,e \tag{20}$$

Comparing the nondiagonal elements in Eqs. (19), we obtain the nondiagonal elements of the diagonal blocks of $\tilde{\alpha}^{(t,1)}$:

$$\tilde{\alpha}_{jl}^{(t,1,k,k)} = \frac{y_{jl}^{(k)}}{\lambda_l^{(2,k)} - \lambda_j^{(2,k)}}$$

$$j \ne l, \qquad j,l = 1,\ldots,m_k, \qquad k = 1,\ldots,e \tag{21}$$

By noting $w_{n_j}^{(0,j)} = 1$, $w_{n_j}^{(1,j)} = 0$, and

$$\boldsymbol{W}^{(1)} = \overset{*}{\boldsymbol{W}}{}^{(1)} + \sum_{k=t+1}^{r}\boldsymbol{U}^{(k,1)}\boldsymbol{\Psi}^{(k)} + \boldsymbol{W}^{(0)}\tilde{\alpha}^{(t,1)} \overset{\text{def}}{=} \hat{\boldsymbol{W}}^{(1)} + \boldsymbol{W}^{(0)}\tilde{\alpha}^{(t,1)}$$

the diagonal elements of $\tilde{\alpha}^{(t,1)}$ can be obtained:

$$\tilde{\alpha}_{jj}^{(t,1)} = -\left[\hat{w}_{n_j}^{(1,j)} + \sum_{\substack{1\le k\le s_t\\ k\ne j}} w_{n_j}^{(0,k)}\tilde{\alpha}_{kj}^{(t,1)}\right], \qquad j = 1,\ldots,s_t \tag{22}$$

Thus $\tilde{\alpha}^{(t,1)}$, $\alpha^{(k,1)}$ $(k = t,\ldots,r)$ and $\boldsymbol{W}^{(1)}$ are determined.

Numerical Example

Consider two 12×12 matrices:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & -2 & 2 & -2 & 2 & -2 & 1 & 1 & -2 & 1 & 0\\ -1 & 3 & -3 & 4 & -4 & 4 & -4 & 2 & 2 & -4 & 2 & 0\\ -1 & 2 & -3 & 6 & -6 & 6 & -6 & 3 & 3 & -6 & 3 & 0\\ -1 & 2 & -4 & 7 & -7 & 8 & -8 & 4 & 4 & -8 & 4 & 0\\ -1 & 2 & -4 & 6 & -7 & 10 & -10 & 5 & 5 & -10 & 5 & 0\\ -1 & 2 & -4 & 6 & -8 & 11 & -11 & 6 & 6 & -12 & 6 & 0\\ -1 & 2 & -4 & 6 & -8 & 10 & -11 & 7 & 7 & -14 & 7 & 0\\ -1 & 2 & -4 & 6 & -8 & 10 & -12 & 7 & 9 & -16 & 8 & 0\\ -1 & 2 & -4 & 6 & -8 & 10 & -12 & 6 & 10 & -17 & 9 & 0\\ -1 & 2 & -4 & 6 & -8 & 10 & -12 & 6 & 9 & -17 & 10 & 0\\ -1 & 2 & -4 & 6 & -8 & 10 & -12 & 6 & 9 & -18 & 10 & 1\\ -1 & 2 & -4 & 6 & -8 & 10 & -12 & 6 & 9 & -18 & 9 & 2 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & -4 & 4 & -3 & 3 & 1 & -1 & -2 & -1 & 3 & 0 & -1\\ 4 & -9 & 8 & -6 & 6 & 2 & -2 & -4 & -2 & 6 & 0 & -2\\ 4 & -11 & 11 & -9 & 9 & 3 & -3 & -6 & -3 & 9 & 0 & -3\\ 4 & -12 & 12 & -11 & 12 & 4 & -4 & -8 & -4 & 12 & 0 & -4\\ 4 & -12 & 12 & -14 & 16 & 5 & -5 & -10 & -5 & 15 & 0 & -5\\ 4 & -12 & 12 & -15 & 16 & 8 & -6 & -12 & -6 & 18 & 0 & -6\\ 4 & -12 & 12 & -15 & 16 & 8 & -5 & -14 & -7 & 21 & 0 & -7\\ 4 & -12 & 12 & -15 & 16 & 8 & -4 & -16 & -8 & 24 & 0 & -8\\ 4 & -12 & 12 & -15 & 16 & 7 & -1 & -19 & -9 & 27 & 0 & -9\\ 4 & -12 & 12 & -15 & 16 & 7 & -1 & -19 & -11 & 30 & 0 & -10\\ 4 & -12 & 12 & -15 & 16 & 7 & -1 & -19 & -13 & 33 & 0 & -11\\ 4 & -12 & 12 & -15 & 16 & 7 & -1 & -19 & -14 & 34 & 1 & -12 \end{bmatrix}$$

All of the eigenvalues of $\boldsymbol{A}$ are 1. In the Jordan canonical form of $\boldsymbol{A}$, there are three blocks of order two and two blocks of order three. We can take

$$\boldsymbol{U}^{(1,1)} = \begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 3 & 3\\ 1 & 3 & 4\\ 1 & 3 & 5\\ 1 & 3 & 5\\ 1 & 3 & 5\\ 1 & 3 & 5\\ 1 & 3 & 5\\ 1 & 3 & 5\\ 1 & 3 & 5\\ 1 & 3 & 5 \end{bmatrix}, \qquad \boldsymbol{U}^{(1,2)} = \begin{bmatrix} 1 & 1 & 1\\ 2 & 2 & 2\\ 2 & 3 & 3\\ 2 & 4 & 4\\ 2 & 4 & 5\\ 2 & 4 & 6\\ 2 & 4 & 6\\ 2 & 4 & 6\\ 2 & 4 & 6\\ 2 & 4 & 6\\ 2 & 4 & 6\\ 2 & 4 & 6 \end{bmatrix}$$

$$\boldsymbol{U}^{(2,1)} = \begin{bmatrix} 1 & 1\\ 2 & 2\\ 3 & 3\\ 4 & 4\\ 5 & 5\\ 6 & 6\\ 7 & 7\\ 7 & 8\\ 7 & 9\\ 7 & 10\\ 7 & 10\\ 7 & 10 \end{bmatrix}, \qquad \boldsymbol{U}^{(2,2)} = \begin{bmatrix} 1 & 1\\ 2 & 2\\ 3 & 3\\ 4 & 4\\ 5 & 5\\ 6 & 6\\ 7 & 7\\ 8 & 8\\ 8 & 9\\ 8 & 10\\ 8 & 11\\ 8 & 11 \end{bmatrix}$$

$$\boldsymbol{U}^{(2,3)} = \begin{bmatrix} 1 & 1\\ 2 & 2\\ 3 & 3\\ 4 & 4\\ 5 & 5\\ 6 & 6\\ 7 & 7\\ 8 & 8\\ 9 & 9\\ 9 & 10\\ 9 & 11\\ 9 & 12 \end{bmatrix}, \qquad \boldsymbol{V}^{(1)} = \begin{bmatrix} -1 & 0 & 0\\ 2 & 0 & 0\\ -1 & -1 & 0\\ 0 & 2 & 0\\ 0 & -1 & -1\\ 0 & 0 & 2\\ 0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}$$

$$
\boldsymbol{V}^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}
$$

For sufficiently small $\varepsilon > 0$ the eigenvalues of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$ are

$$
1 + \sqrt{\varepsilon} - \varepsilon, \qquad 1 + \sqrt{\varepsilon} + \varepsilon, \qquad 1 + \sqrt{\varepsilon} + 2\varepsilon
$$

$$
1 - \sqrt{\varepsilon} - \varepsilon, \qquad 1 - \sqrt{\varepsilon} + \varepsilon, \qquad 1 - \sqrt{\varepsilon} + 2\varepsilon
$$

$$
1 + \rho_1,\ 1 + \rho_2; \qquad 1 + \rho_3,\ 1 + \rho_4; \qquad 1 + \rho_5,\ 1 + \rho_6
$$

In the preceding expressions

$$
\rho_1 = \xi_1^+ + \xi_1^-, \qquad \rho_2 = \xi_2^+ + \xi_2^-, \qquad \rho_3 = \omega_+\xi_1^+ + \omega_-\xi_1^-
$$

$$
\rho_4 = \omega_+\xi_2^+ + \omega_-\xi_2^-, \qquad \rho_5 = \omega_-\xi_1^+ + \omega_+\xi_1^-
$$

$$
\rho_6 = \omega_-\xi_2^+ + \omega_+\xi_2^-
$$

where

$$
\omega_\pm = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}i, \qquad \xi_1^\pm = \left\{\varepsilon/2 \pm \left[(\varepsilon/2)^2 + (2\varepsilon/3)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}}
$$

$$
\xi_2^\pm = \left\{\varepsilon/2 \pm \left[(\varepsilon/2)^2 - (2\varepsilon/3)^3\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}}
$$

It is easy to see that the first- to third-order derivatives with respect to $\eta = \varepsilon^{1/2}$ of the eigenvalues in the first and second groups are $(1, 1, 1; -2, 2, 4; 0, 0, 0)$ and $(-1, -1, -1; -2, 2, 4; 0, 0, 0)$, respectively. The first- to third-order derivatives with respect to $\eta = \varepsilon^{1/3}$ of eigenvalues in the third to fifth groups are $(1, 1; -\frac{4}{3}, \frac{4}{3}; 0, 0)$, $(\omega_+, \omega_+; -\frac{4}{3}\omega_-, \frac{4}{3}\omega_-; 0, 0)$, $(\omega_-, \omega_-; -\frac{4}{3}\omega_+, \frac{4}{3}\omega_+; 0, 0)$, respectively. The calculations of this paper are conducted with 16 significant decimal digits. The calculated first- to third-order derivatives of the eigenvalues are accurate in at least 15 significant digits except the third-order derivatives of eigenvalues in the fourth and fifth groups where the calculated results are not the exact values $(0, 0)$ and $(0, 0)$ but $\approx(3 \times 10^{-9}, 3 \times 10^{-8})$, $(3 \times 10^{-9}, 3 \times 10^{-8})$, respectively. With at least 15 significant digits the calculated differentiable eigenvectors and their first derivatives are

$$
\boldsymbol{W}_1^{(0)} = \boldsymbol{W}_2^{(0)} = \frac{1}{15}\begin{bmatrix} 15 & 5 & 3 \\ 15 & 10 & 6 \\ 15 & 15 & 9 \\ 15 & 15 & 12 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \\ 15 & 15 & 15 \end{bmatrix}
$$

$$
\boldsymbol{W}_3^{(0)} = \boldsymbol{W}_4^{(0)} = \boldsymbol{W}_5^{(0)} = \frac{1}{7}\begin{bmatrix} 1 & 0.7 \\ 2 & 1.4 \\ 3 & 2.1 \\ 4 & 2.8 \\ 5 & 3.5 \\ 6 & 4.2 \\ 7 & 4.9 \\ 7 & 5.6 \\ 7 & 6.3 \\ 7 & 7 \\ 7 & 7 \\ 7 & 7 \end{bmatrix}
$$

$$
\boldsymbol{W}_1^{(1)} = \frac{1}{15}\begin{bmatrix} 0 & 0 & 0 \\ 15 & 0 & 0 \\ 15 & 0 & 0 \\ 15 & 5 & 0 \\ 15 & 5 & 0 \\ 15 & 5 & 3 \\ 15 & 5 & 3 \\ 15 & 5 & 3 \\ 15 & 5 & 3 \\ 15 & 5 & 3 \\ 15 & 5 & 3 \\ 15 & 5 & 3 \end{bmatrix}, \qquad \boldsymbol{W}_3^{(1)} = \frac{1}{7}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0.7 \\ 1 & 0.7 \end{bmatrix}
$$

$$
\boldsymbol{W}_2^{(1)} = -\boldsymbol{W}_1^{(1)}, \qquad \boldsymbol{W}_4^{(1)} = \omega_+\boldsymbol{W}_3^{(1)}, \qquad \boldsymbol{W}_5^{(1)} = \omega_-\boldsymbol{W}_3^{(1)}
$$

It is easy to see that the exact solution of eigenvectors of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$ are

$$
\boldsymbol{W}_1(\eta) = \frac{1}{15(1+\eta)}\begin{bmatrix} 15 & 5(1+\eta) & 3(1+\eta) \\ 15(1+2\eta) & 10(1+\eta) & 6(1+\eta) \\ 15(1+2\eta) & 15(1+\eta) & 9(1+\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 12(1+\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 15(1+\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \\ 15(1+2\eta) & 5(3+4\eta) & 3(5+6\eta) \end{bmatrix}
$$

$$
\boldsymbol{W}_2(\eta) = \boldsymbol{W}_1(-\eta), \qquad \boldsymbol{W}_3 = \frac{1}{7}\begin{bmatrix} 1 & 0.7 \\ 2 & 1.4 \\ 3 & 2.1 \\ 4 & 2.8 \\ 5 & 3.5 \\ 6 & 4.2 \\ 7 & 4.9 \\ \sigma_1 & 5.6 \\ \sigma_2 & 6.3 \\ \sigma_2 & 7 \\ \sigma_2 & 7\sigma_3 \\ \sigma_2 & 7\sigma_4 \end{bmatrix}
$$

$$W_4 = \frac{1}{7}\begin{bmatrix} 1 & 0.7 \\ 2 & 1.4 \\ 3 & 2.1 \\ 4 & 2.8 \\ 5 & 3.5 \\ 6 & 4.2 \\ 7 & 4.9 \\ \sigma_5 & 5.6 \\ \sigma_6 & 6.3 \\ \sigma_6 & 7 \\ \sigma_6 & 7\sigma_7 \\ \sigma_6 & 7\sigma_8 \end{bmatrix}, \qquad W_5 = \frac{1}{7}\begin{bmatrix} 1 & 0.7 \\ 2 & 1.4 \\ 3 & 2.1 \\ 4 & 2.8 \\ 5 & 3.5 \\ 6 & 4.2 \\ 7 & 4.9 \\ \sigma_9 & 5.6 \\ \sigma_{10} & 6.3 \\ \sigma_{10} & 7 \\ \sigma_{10} & 7\sigma_{11} \\ \sigma_{10} & 7\sigma_{12} \end{bmatrix}$$

In the expressions of $W_1(\eta)$ and $W_2(\eta)$, $\eta = \varepsilon^{1/2}$. In the expressions of $W^{(3)}$, $W^{(4)}$, and $W^{(5)}$,

$$\sigma_1 = \frac{7 + 8(\rho_1 + \varepsilon + \rho_1^2)}{1 + \rho_1 + \varepsilon + \rho_1^2}, \qquad \sigma_2 = \frac{7 + 8\rho_1 + 9(\varepsilon + \rho_1^2)}{1 + \rho_1 + \varepsilon + \rho_1^2}$$

$$\sigma_3 = \frac{10 + 11(\rho_2 + \varepsilon + \rho_2^2)}{10(1 + \rho_2 + \varepsilon + \rho_2^2)}, \qquad \sigma_4 = \frac{10 + 11\rho_2 + 12(\varepsilon + \rho_2^2)}{10(1 + \rho_2 + \varepsilon + \rho_2^2)}$$

$$\sigma_5 = \frac{7 + 8(\rho_3 + \varepsilon + \rho_3^2)}{1 + \rho_3 + \varepsilon + \rho_3^2}, \qquad \sigma_6 = \frac{7 + 8\rho_3 + 9(\varepsilon + \rho_3^2)}{1 + \rho_3 + \varepsilon + \rho_3^2}$$

$$\sigma_7 = \frac{10 + 11(\rho_4 + \varepsilon + \rho_4^2)}{10(1 + \rho_4 + \varepsilon + \rho_4^2)}, \qquad \sigma_8 = \frac{10 + 11\rho_4 + 12(\varepsilon + \rho_4^3)}{10(1 + \rho_4 + \varepsilon + \rho_4^2)}$$

$$\sigma_9 = \frac{7 + 8(\rho_5 + \varepsilon + \rho_5^2)}{1 + \rho_5 + \varepsilon + \rho_5^2}, \qquad \sigma_{10} = \frac{7 + 8\rho_5 + 9(\varepsilon + \rho_5^2)}{1 + \rho_5 + \varepsilon + \rho_5^2}$$

$$\sigma_{11} = \frac{10 + 11(\rho_6 + \varepsilon + \rho_6^2)}{10(1 + \rho_6 + \varepsilon + \rho_6^2)}, \qquad \sigma_{12} = \frac{10 + 11\rho_6 + 12(\varepsilon + \rho_6^2)}{10(1 + \rho_6 + \varepsilon + \rho_6^2)}$$

In the preceding expressions $W_k(\eta)$, $W_k^{(0)}$, $W_k^{(1)}$ denote the matrix of eigenvectors of $A + \epsilon B$, the matrix of differentiable eigenvectors and the matrix of first-order eigenvector derivatives respectively associated with the kth group of eigenvalues of $A + \epsilon B$. By direct checking it is easy to see that the calculated $W^{(0)}$ and $W^{(1)}$ are correct.

Conclusions

This paper gives a closed-form solution to the first- to third-order perturbation coefficients of the eigenvalues and first-order perturbation coefficients of eigenvectors of a defective matrix on the assumptions Eqs. (2) and the assumption that for any k $(1 \le k \le e)$ the eigenvalues of problem (13) are all simple. Under these assumptions the higher-order perturbation coefficients of eigenvalues and eigenvectors can be calculated from the higher-order perturbation equations by use of the similar way, but the expressions for the solutions will be more complicated.

If for some k problem (13) has repeated eigenvalues, then the perturbation coefficients of eigenvalues and eigenvectors can be calculated by partitioning the matrices into smaller blocks and by use of the similar techniques. In this case, however, the work for the solutions will be much more elaborate. For example, in order to calculate $W^{(0)}$ and $W^{(1)}$ at least the equation for $W^{(\nu+2)}$ and that for $W^{(\nu+3)}$ must be used, respectively.

If Eqs. (2) do not hold, we do not know what will happen to the perturbed eigenvalue problem, and we do not know how to calculate the perturbation coefficients of eigenvalues and eigenvectors. In this case, the perturbed eigenvalue problem will perhaps be defective under some conditions and needs to find some new way to analyze the eigensensitivity.

Appendix: Proof of the Nonsingularity of $\Lambda^{(1)}$

If $\Lambda^{(1)}$ is singular, then it must have a diagonal element $\lambda_l^{(1)} = 0$. Comparing the lth column of Eqs. (6), we obtain

$$\sum_{k=t}^{r} Q^{(j,k,1)} \alpha_l^{(k,0)} = 0, \qquad j = t, \ldots, r \tag{A1}$$

where $\alpha_l^{(k,0)}$ is the lth column of $\alpha^{(k,0)}$ $(k = t, \ldots, r)$. Equations (2) ensure the nonsingularity of the coefficient matrix of Eqs. (A1), a system of linear homogeneous algebraic equations for $\alpha_l^{(k,0)}$ $(k = t, \ldots, r)$. Therefore the system only has zero solution, which contradicts the fact that $\alpha_l^{(t,0)}$ is an eigenvector of problem (8).

Acknowledgment

The authors would like to thank Wen-liang Wang of our university for his advice.

References

[1]Mills-Curran, W. C., "Calculation of Eigenvectors Derivatives for Structures with Repeated Eigenvalues," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 867–871.

[2]Mills-Curran, W. C., "Comment on Eigenvector Derivatives with Repeated Eigenvalues," *AIAA Journal*, Vol. 28, No. 10, 1990, pp. 1846–1846.

[3]Dailey, R. L., "Eigenvector Derivatives with Repeated Eigenvalues," *AIAA Journal*, Vol. 27, No. 4, 1989, pp. 486–491.

[4]Felszeghy, S. F., "Development of Biorthonormal Eigenvectors for Modal Analysis of Linear Discrete Non-Classically Damped Systems," *Journal of Sound and Vibration*, Vol. 176, No. 2, 1994, pp. 255–269.

[5]Luongo, A., "Free Vibration and Sensitivity Analysis of a Defective Two Degree-of-Freedom System," *AIAA Journal*, Vol. 33, No. 1, 1995, pp. 120–127.

[6]Luongo, A., "Eigensolutions Sensitivity for Nonsymmetric Matrices with Repeated Eigenvalues," *AIAA Journal*, Vol. 31, No. 7, 1993, pp. 1321–1328.

A. Berman
Associate Editor